Quantum Entanglement, Interaction, and the Classical Limit

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Two or more quantum systems are said to be in an entangled or non-factorisable state if their joint (supposedly pure) wave-function is not expressible as a product of individual wave functions but is instead a superposition of product states. Only when the systems are in a factorisable state they can be considered to be separated (in the sense of Bell). We show that whenever two quantum systems interact with each other, it is impossible that all factorisable states remain factorisable during the interaction unless the full Hamiltonian does not couple these systems so to say unless they do not really interact. We also present certain conditions under which particular factorisable states remain factorisable although they represent a bipartite system whose components mutually interact. We identify certain quasi-classical regimes that satisfy these conditions and show that they correspond to classical, pre-quantum, paradigms associated to the concept of particle. – PACS number: O3.65.Bz

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1. Introduction

The term entanglement was first introduced by Schrödinger who described this as the characteristic trait of quantum mechanics, "the one that enforces its entire departure from classical lines of thought" [1]. Bell's inequalities [2] show that when two systems are prepared in an entangled state, the knowledge of the whole cannot be reduced to the knowledge of the parts, and that to some extent the systems lose their individuality. It is only when their joint wave-function is factorisable that they are separable[1]. It is therefore interesting to investigate which are the situations such that two systems, initially prepared in a (pure) product state, remain in such a state although they mutually interact. We shall show that, when the Hilbert spaces associated to the interacting systems A and B are finite dimensional, if we impose that all the product states remain product states during the interaction, the full Hamiltonian can be factorised as follows: $H_{\mathrm{AB}}(t) = H_{\mathrm{A}}(t) \otimes I_{\mathrm{B}} + I_{\mathrm{A}} \otimes H_{\mathrm{B}}(t)$, where $H_i(t)$ acts on the "i" system only while I_j is the identity operator on the "j" system ($i, j = \mathrm{A}, \mathrm{B}$). In other words, in quantum mechanics there is no interaction without entanglement. We shall also present two conditions under which particular factorisable (non-necessary pure) states remain factorisable during the interaction. Finally we shall show that, when two particles obey the non-relativistic Schrödinger equation, we can distinguish three regimes for which this condition is satisfied, that correspond to the following classical paradigms: material point, test-particle, and diluted particle (droplet model). We reinterpret these results in the light of the "predictability sieve" criterion (PS criterion) proposed by Zurek in the decoherence approach [5 – 7] and note that, although they are obtained in an oversimplified quantum modellisation of a system and an environment (here, the universe consists of two particles considered in first quantisation that interact through a position-dependent potential in the non-relavistic regime), these results confirm the basic intuition of the PS criterion: the classical behavior corresponds to islands of the Hilbert space characterized by the minimal Shannon-von Neumann entropy (or equivalently by the maximal coherence of the reduced state of the system obtained after tracing out the environment).

2. Two Interacting Finite-Dimensional Systems: Entanglement Versus Interaction

Let us consider two interacting quantum systems A and B. We assume that the Hilbert spaces associated

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[1] It can be shown that, whenever two distant systems are in an entangled (pure) state, there exist well-chosen observables such that the associated correlations do not admit a local realistic explanation, which is revealed by the violation of well-chosen Bell's inequalities [3, 4].

to these systems are finite dimensional (of dimensions $\mathrm{d_A}$ and $\mathrm{d_B}$, respectively), that the wave-function of the full system is a pure state of $\boldsymbol{C}^{\mathrm{d_A}} \otimes \boldsymbol{C}^{\mathrm{d_B}}$ and obeys the Schrödinger equation

$$i\hbar\partial_t \boldsymbol{\Psi}_{\mathrm{AB}}(t) = H_{\mathrm{AB}}(t)\boldsymbol{\Psi}_{\mathrm{AB}}(t), \tag{1}$$

where $H_{\mathrm{AB}}(t)$ is a self-adjoint operator that acts on $\boldsymbol{C}^{\mathrm{d_A}} \otimes \boldsymbol{C}^{\mathrm{d_B}}$, that we assume to be sufficiently regular in time in order to ensure that the temporal Taylor development of the wave-function is valid up to the second order in time.

Main Theorem

All the product states remain product states during the interaction if and only if the full Hamiltonian can be factorised as follows:

$$H_{\mathrm{AB}}(t) = H_{\mathrm{A}}(t) \otimes I_{\mathrm{B}} + I_{\mathrm{A}} \otimes H_{\mathrm{B}}(t), \tag{2}$$

where H_i acts on the ith system only while I_j is the identity operator on the jth system ($i, j = \mathrm{A}, \mathrm{B}$).

In order to prove this theorem, we shall firstly prove the following lemma:

Lemma

A pure product state remains product state during the interaction if and only if, during its evolution, the Hamiltonian never couples this product state to a product state that is bi-orthogonal to it.

Proof of the Lemma

A) Proof of the necessary condition. Let us assume that a state initially (at time t) factorisable remains a product state throughout the evolution: $\boldsymbol{\Psi}_{\mathrm{AB}}(t') = \psi_{\mathrm{A}}(t') \otimes \psi_{\mathrm{B}}(t'), \forall t' \geq t$. Then, in virtue of the Leibnitz rule for the time derivative of a product, we get that $H_{\mathrm{AB}}(t')\boldsymbol{\Psi}_{\mathrm{AB}}(t') = i\hbar\partial_{t'}\boldsymbol{\Psi}_{\mathrm{AB}}(t') = i\hbar(\partial_{t'}(\psi_{\mathrm{A}}(t')) \otimes \psi_{\mathrm{B}}(t') + \psi_{\mathrm{A}}(t') \otimes \partial_{t'}(\psi_{\mathrm{B}}(t')))$. The former state is obviously not biorthogonal to $\psi_{\mathrm{A}}(t') \otimes \psi_{\mathrm{B}}(t') = \boldsymbol{\Psi}_{\mathrm{AB}}(t')$; actually it belongs to the space which is orthogonal to the space that contains the states biorthogonal to $\boldsymbol{\Psi}_{\mathrm{AB}}(t')$. In the appendix we show that, when a state is factorisable, the squared lenght of the component of $H_{\mathrm{AB}}(t')\boldsymbol{\Psi}_{\mathrm{AB}}(t')$ that is biorthogonal to this state is proportional to the rate of decrease of the trace of the squared reduced density matrix, where the reduced density matrix is obtained from the full density matrix after tracing out one of the subsystems (A or B). This rate is also equal to the linear entropy production, where the linear entropy of the reduced state ρ is defined as $\mathrm{Tr}\rho - \mathrm{Tr}\rho^2 = 1 - \mathrm{Tr}\rho^2$. It is a good measure of the degree of entanglement between A and B, when the full state is pure, and nearly factorisable (it coincides with the linear term in the Taylor development of the Shannon-von Neumann entropy of the reduced density matrix, the constant term being equal to zero for factorisable states). Formulated so, the physical meaning of the necessary condition is very trivial: if the full state remains factorisable throughout time, then no entanglement occurs (the rate of creation of the entanglement is equal to zero).

B) Proof of the sufficient condition. Let us consider that at time t the system is prepared in a product state $\boldsymbol{\Psi}_{\mathrm{AB}}(t) = \psi_{\mathrm{A}}(t) \otimes \psi_{\mathrm{B}}(t)$. Let us choose a basis of product states $|\psi_{\mathrm{A}}^i\rangle \otimes |\phi_{\mathrm{B}}^j\rangle$ similar to the basis introduced in the appendix, so to say a basis of product states $|\psi_{\mathrm{A}}^i\rangle \otimes |\phi_{\mathrm{B}}^j\rangle$ ($i : 1 \ldots \mathrm{d_A}; j : 1 \ldots \mathrm{d}_B$, and $\langle \psi_{\mathrm{A}}^i | \psi_{\mathrm{A}}^{i'}\rangle = \delta_{ii'}; \langle \phi_{\mathrm{B}}^j | \phi_{\mathrm{B}}^{j'}\rangle = \delta_{jj'}$) such that $\psi_{\mathrm{A}}(t) = |\psi_{\mathrm{A}}^1\rangle$ and $\phi_{\mathrm{B}}(t) = |\phi_{\mathrm{A}}^1\rangle$. Let us define the Hamiltonian by evaluating its matricial elements in this basis as follows:

$$H_{ikjl} = \langle \psi_{\mathrm{A}}^i| \otimes \langle \phi_{\mathrm{B}}^j| H_{\mathrm{AB}}(t) |\psi_{\mathrm{A}}^k\rangle \otimes |\phi_{\mathrm{B}}^l\rangle. \tag{3}$$

When the Hamiltonian does not couple $\boldsymbol{\Psi}_{\mathrm{AB}}(t)$ to states that are bi-orthogonal to it, $\Sigma_{i:2\ldots\mathrm{d_A}; j:2\ldots\mathrm{d_B}} |H_{i1j1}|^2 = 0$ (where H_{ikjl} is defined in (3) and, in virtue of (1):

$$i\hbar\partial_t \boldsymbol{\Psi}_{\mathrm{AB}}(t) = H_{\mathrm{AB}}(t)\boldsymbol{\Psi}_{\mathrm{AB}}(t) = \tag{4}$$

$$(\Sigma_{i:1\ldots\mathrm{d_A}} H_{i111}|\psi_{\mathrm{A}}^i\rangle) \otimes |\phi_{\mathrm{B}}^1\rangle + |\psi_{\mathrm{A}}^1\rangle(\Sigma_{j:2\ldots\mathrm{d_B}} H_{11j1}|\phi_{\mathrm{B}}^j\rangle).$$

We can rewrite this equation as follows:

$$\begin{aligned} i\hbar\partial_t \boldsymbol{\Psi}_{\mathrm{AB}}(t) = {} & (H_{\mathrm{A}}^{\mathrm{eff.}}(t) \cdot \psi_{\mathrm{A}}(t)) \otimes \psi_{\mathrm{B}}(t) \\ & + \psi_{\mathrm{A}}(t) \otimes (H_{\mathrm{B}}^{\mathrm{eff.}}(t) \cdot \psi_{\mathrm{B}}(t)), \end{aligned} \tag{5}$$

where the effective Hamiltonians $H^{\mathrm{eff.}}$ are defined:

$$H_{\mathrm{A}}^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{B}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t)) \tag{6}$$

and

$$\begin{aligned} H_B^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{B}}(t) = {} & \mathrm{Tr}_{\mathrm{A}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t)) \\ & - (\mathrm{Tr}_{\mathrm{AB}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t))) \cdot \rho_{\mathrm{B}}(t). \end{aligned} \tag{7}$$

In these expressions Tr_i represents the partial trace over the degrees of freedom assigned to the system i, while $\rho_{\mathrm{AB}}(t)$ is the projector onto $\boldsymbol{\Psi}_{\mathrm{AB}}(t)$, $\rho_A(t) =$

$\mathrm{Tr}_\mathrm{B}\rho_\mathrm{AB}(t)$, and $\rho_\mathrm{B}(t) = \mathrm{Tr}_\mathrm{A}\rho_\mathrm{AB}(t)$. For instance, we have that

$$\begin{aligned}
&\mathrm{Tr}_\mathrm{B}(H_\mathrm{AB}(t)\rho_\mathrm{AB}(t))\\
&= \Sigma_{l:1\ldots\mathrm{d_B}}\langle\phi^l_\mathrm{B}|H_\mathrm{AB}|\psi^1_\mathrm{A}\rangle\otimes|\phi^1_\mathrm{B}\rangle\langle\psi^1_\mathrm{A}|\otimes\langle\phi^1_\mathrm{B}|\phi^l_\mathrm{B}\rangle\\
&= \Sigma_{l:1\ldots\mathrm{d_B}}\langle\phi^l_\mathrm{B}|\Sigma_{i:1\ldots\mathrm{d_A},j:1\ldots\mathrm{d_B}}H_{i1j1}|\psi^i_\mathrm{A}\rangle\otimes|\phi^j_\mathrm{B}\rangle\delta_{l1}\langle\psi^1_\mathrm{A}|\\
&= \Sigma_{l:1\ldots\mathrm{d_B}}\Sigma_{i:1\ldots\mathrm{d_A},j:1\ldots\mathrm{d_B}}H_{i1j1}|\psi^i_\mathrm{A}\rangle\delta_{lj}\delta_{l1}\langle\psi^1_\mathrm{A}|\\
&= (\Sigma_{i:1\ldots\mathrm{d_A}}H_{i111}|\psi^i_\mathrm{A}\rangle\langle\psi^1_\mathrm{A}|),
\end{aligned}$$

so that $H^{\mathrm{eff.}}_\mathrm{A}(t)\cdot\psi_\mathrm{A}(t) = \Sigma_{i:1\ldots\mathrm{d_A}}H_{i111}|\psi^i_\mathrm{A}\rangle$. Similarly we get that $H^{\mathrm{eff.}}_\mathrm{B}(t)\cdot\psi_\mathrm{B}(t) = \Sigma_{i:2\ldots\mathrm{d_B}}H_{1i11}|\psi^i_\mathrm{B}\rangle$.

Let us consider the product state $\psi^{\mathrm{red}}_\mathrm{A}(t')\otimes\psi^{\mathrm{red}}_\mathrm{B}(t')$, where $\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t')$ is a solution of the reduced Schrödinger equation $i\hbar\partial_{t'}\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t') = H^{\mathrm{eff.}}_{\mathrm{A(B)}}(t')\cdot\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t')$ for the initial condition $\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t) = \psi_{\mathrm{A(B)}}(t)$. Obviously, $i\hbar\partial_{t'}\psi^{\mathrm{red}}_\mathrm{A}(t')\otimes\psi^{\mathrm{red}}_\mathrm{B}(t') = H_\mathrm{AB}\psi^{\mathrm{red}}_\mathrm{A}(t')\otimes\psi^{\mathrm{red}}_\mathrm{B}(t')$ and $\boldsymbol{\Psi}_\mathrm{AB}(t) = \psi^{\mathrm{red}}_\mathrm{A}(t)\otimes\psi^{\mathrm{red}}_\mathrm{B}(t)$ so that, in virtue of the deterministic character of Schrödinger's equation, $\boldsymbol{\Psi}_\mathrm{AB}(t') = \psi^{\mathrm{red}}_\mathrm{A}(t')\otimes\psi^{\mathrm{red}}_\mathrm{B}(t')$, $\forall t'\geq t$ which ends the proof of the lemma.

We shall now prove the main theorem.

Proof of the Main Theorem

A) Proof of the necessary condition. Let us choose a basis of product states $|\psi^i_\mathrm{A}\rangle\otimes|\phi^j_\mathrm{B}\rangle$ $(i:1\ldots\mathrm{d_A}; j:1\ldots\mathrm{d_B}$ and $\langle\psi^i_\mathrm{A}|\psi^j_\mathrm{A}\rangle = \delta_{ij} = \langle\phi^i_\mathrm{B}|\phi^j_\mathrm{B}\rangle)$. If we impose that all the product states remain product states during the interaction, then, in virtue of the lemma, the full Hamiltonian never couples a product state to a product state that is bi-orthogonal to it. Then, at any time t, $\Sigma_{i:2\ldots\mathrm{d_A};j:2\ldots\mathrm{d_B}}|H_{i1j1}|^2 = 0$ (where H_{ikjl} is defined in (3)) so that we have

$$\begin{aligned}
&H_\mathrm{AB}(t)\cdot|\psi^i_\mathrm{A}\rangle\otimes|\phi^j_\mathrm{B}\rangle =\\
&\quad|\Delta^{ij}_\mathrm{A}\psi^i_\mathrm{A}\rangle\otimes|\phi^j_\mathrm{B}\rangle + |\psi^i_\mathrm{A}\rangle\otimes|\Delta^{ij}_\mathrm{B}\phi^j_\mathrm{B}\rangle
\end{aligned}$$

where

$$|\Delta^{ij}_\mathrm{A}\psi^i_\mathrm{A}\rangle = \Sigma_{k:1\ldots\mathrm{d_A}}H_{kijj}|\psi^k_\mathrm{A}\rangle \tag{8}$$

and

$$|\Delta^{ij}_\mathrm{B}\phi^i_\mathrm{B}\rangle = \Sigma_{k:1\ldots\mathrm{d_B},k\neq j}H_{iikj}|\phi^k_\mathrm{B}\rangle. \tag{9}$$

Let us consider that at the time t the system is prepared along one of the first four states $\boldsymbol{\Psi}^i_\mathrm{AB}$ $(i:1,\ldots 4)$ of this basis: $\boldsymbol{\Psi}^1_\mathrm{AB}(t) = |\psi^1_\mathrm{A}\rangle\otimes|\phi^1_\mathrm{B}\rangle$, $\boldsymbol{\Psi}^2_\mathrm{AB}(t) = |\psi^1_\mathrm{A}\rangle\otimes|\phi^2_\mathrm{B}\rangle$, $\boldsymbol{\Psi}^3_\mathrm{AB}(t) = |\psi^2_\mathrm{A}\rangle\otimes|\phi^1_\mathrm{B}\rangle$, $\boldsymbol{\Psi}^4_\mathrm{AB}(t) = |\psi^2_\mathrm{A}\rangle\otimes|\phi^2_\mathrm{B}\rangle$. Then

$$\begin{aligned}
H_\mathrm{AB}(t)\cdot\boldsymbol{\Psi}^1_\mathrm{AB}(t) &= |\Delta^{11}_\mathrm{A}\psi^1_\mathrm{A}\rangle\otimes|\phi^1_\mathrm{B}\rangle + |\psi^1_\mathrm{A}\rangle\otimes|\Delta^{11}_\mathrm{B}\phi^1_\mathrm{B}\rangle,\\
H_\mathrm{AB}(t)\cdot\boldsymbol{\Psi}^2_\mathrm{AB}(t) &= |\Delta^{12}_\mathrm{A}\psi^1_\mathrm{A}\rangle\otimes|\phi^2_\mathrm{B}\rangle + |\psi^1_\mathrm{A}\rangle\otimes|\Delta^{12}_\mathrm{B}\phi^2_\mathrm{B}\rangle,\\
H_\mathrm{AB}(t)\cdot\boldsymbol{\Psi}^3_\mathrm{AB}(t) &= |\Delta^{21}_\mathrm{A}\psi^2_\mathrm{A}\rangle\otimes|\phi^1_\mathrm{B}\rangle + |\psi^2_\mathrm{A}\rangle\otimes|\Delta^{21}_\mathrm{B}\phi^1_\mathrm{B}\rangle,\\
H_\mathrm{AB}(t)\cdot\boldsymbol{\Psi}^4_\mathrm{AB}(t) &= |\Delta^{22}_\mathrm{A}\psi^2_\mathrm{A}\rangle\otimes|\phi^2_\mathrm{B}\rangle + |\psi^2_\mathrm{A}\rangle\otimes|\Delta^{22}_\mathrm{B}\phi^2_\mathrm{B}\rangle,
\end{aligned}$$

By linearity

$$\begin{aligned}
&H_\mathrm{AB}(t)\cdot\frac{1}{\sqrt{2}}(\boldsymbol{\Psi}^1_\mathrm{AB}(t)+\boldsymbol{\Psi}^3_\mathrm{AB}(t))\\
&\quad= H_\mathrm{AB}(t)\cdot\frac{1}{\sqrt{2}}(|\psi^1_\mathrm{A}\rangle+|\psi^2_\mathrm{A}\rangle)\otimes|\phi^1_\mathrm{B}\rangle\\
&\quad= \frac{1}{\sqrt{2}}((|\Delta^{11}_\mathrm{A}\psi^1_\mathrm{A}\rangle+|\Delta^{21}_\mathrm{A}\psi^2_\mathrm{A}\rangle)\otimes|\phi^1_\mathrm{B}\rangle\\
&\qquad\qquad+|\psi^1_\mathrm{A}\rangle\otimes|\Delta^{11}_\mathrm{B}\phi^1_\mathrm{B}\rangle+|\psi^2_\mathrm{A}\rangle\otimes|\Delta^{21}_\mathrm{B}\phi^1_\mathrm{B}\rangle)\\
&\quad= \frac{1}{\sqrt{2}}((|\Delta^{11}_\mathrm{A}\psi^1_\mathrm{A}\rangle+|\Delta^{21}_\mathrm{A}\psi^2_\mathrm{A}\rangle)\otimes|\phi^1_\mathrm{B}\rangle\\
&\qquad\qquad+\frac{1}{2}(|\psi^1_\mathrm{A}\rangle+|\psi^2_\mathrm{A}\rangle)\otimes(|\Delta^{11}_\mathrm{B}\phi^1_\mathrm{B}\rangle+|\Delta^{21}_\mathrm{B}\phi^1_\mathrm{B}\rangle)\\
&\qquad+\frac{1}{2}(|\psi^1_\mathrm{A}\rangle-|\psi^2_\mathrm{A}\rangle)\otimes(|\Delta^{11}_\mathrm{B}\phi^1_\mathrm{B}\rangle-|\Delta^{21}_\mathrm{B}\phi^1_\mathrm{B}\rangle),
\end{aligned}$$

$\frac{1}{\sqrt{2}}(|\psi^1_\mathrm{A}\rangle-|\psi^2_\mathrm{A}\rangle)$ is orthogonal to $\frac{1}{\sqrt{2}}(|\psi^1_\mathrm{A}\rangle+|\psi^2_\mathrm{A}\rangle)$, so that $H_\mathrm{AB}(t)\cdot\frac{1}{\sqrt{2}}(\boldsymbol{\Psi}^1_\mathrm{AB}(t)+\boldsymbol{\Psi}^3_\mathrm{AB}(t))$ couples $\frac{1}{\sqrt{2}}(\boldsymbol{\Psi}^1_\mathrm{AB}(t)+\boldsymbol{\Psi}^3_\mathrm{AB}(t))$ to a bi-orthogonal state unless $(|\Delta^{11}_\mathrm{B}\phi^1_\mathrm{B}\rangle-|\Delta^{21}_\mathrm{B}\phi^1_\mathrm{B}\rangle)$ is parallel to $|\phi^1_\mathrm{B}\rangle$. Now, $\frac{1}{\sqrt{2}}(\boldsymbol{\Psi}^1_\mathrm{AB}(t)+\boldsymbol{\Psi}^3_\mathrm{AB}(t))$ is a product state so that, in virtue of the lemma, the following constraint must be satisfied:

$$(|\Delta^{11}_\mathrm{B}\phi^1_\mathrm{B}\rangle-|\Delta^{21}_\mathrm{B}\phi^1_\mathrm{B}\rangle) = \lambda|\phi^1_\mathrm{B}\rangle.$$

The same reasoning is valid with the states $\frac{1}{\sqrt{2}}(\boldsymbol{\Psi}^2_\mathrm{AB}(t)+\boldsymbol{\Psi}^4_\mathrm{AB}(t))$, $\frac{1}{\sqrt{2}}(\boldsymbol{\Psi}^1_\mathrm{AB}(t)+\boldsymbol{\Psi}^2_\mathrm{AB}(t))$ and $\frac{1}{\sqrt{2}}(\boldsymbol{\Psi}^3_\mathrm{AB}(t)+\boldsymbol{\Psi}^4_\mathrm{AB}(t))$ and leads to the following constraints:

$$\begin{aligned}
&(|\Delta^{12}_\mathrm{B}\phi^2_\mathrm{B}\rangle-|\Delta^{22}_\mathrm{B}\phi^2_\mathrm{B}\rangle) = \lambda'|\phi^2_\mathrm{B}\rangle,\\
&(|\Delta^{11}_\mathrm{A}\psi^1_\mathrm{A}\rangle-|\Delta^{12}_\mathrm{A}\psi^1_\mathrm{A}\rangle) = \lambda''|\psi^1_\mathrm{A}\rangle,\\
&(|\Delta^{21}_\mathrm{A}\psi^2_\mathrm{A}\rangle-|\Delta^{22}_\mathrm{A}\psi^2_\mathrm{A}\rangle) = \lambda'''|\psi^2_\mathrm{A}\rangle.
\end{aligned}$$

By the definition (9), $|\Delta^{ij}_\mathrm{B}\phi^j_\mathrm{B}\rangle$ is orthogonal to $|\phi^j_\mathrm{B}\rangle$ so that necessarily $\lambda = \lambda' = 0$. Let us now consider the product state $(\boldsymbol{\Psi}^1_\mathrm{AB}(t)+\boldsymbol{\Psi}^2_\mathrm{AB}(t)+\boldsymbol{\Psi}^3_\mathrm{AB}(t)+\boldsymbol{\Psi}^4_\mathrm{AB}(t))$.

By linearity

$$
\begin{aligned}
&H_{\mathrm{AB}}(t)\cdot\frac{1}{2}(\boldsymbol{\Psi}^1_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^2_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^3_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^4_{\mathrm{AB}}(t))\\
&\quad=H_{\mathrm{AB}}(t)\cdot\frac{1}{2}(|\psi^1_{\mathrm{A}}\rangle+|\psi^2_{\mathrm{A}}\rangle)\otimes(|\phi^1_{\mathrm{B}}\rangle+|\phi^2_{\mathrm{B}}\rangle)\\
&\quad=\frac{1}{2}((|\Delta^{11}_{\mathrm{A}}\psi^1_{\mathrm{A}}\rangle+|\Delta^{21}_{\mathrm{A}}\psi^2_{\mathrm{A}}\rangle)\otimes|\phi^1_{\mathrm{B}}\rangle+(|\Delta^{12}_{\mathrm{A}}\psi^1_{\mathrm{A}}\rangle\\
&\qquad+|\Delta^{22}_{\mathrm{A}}\psi^2_{\mathrm{A}}\rangle)\otimes|\phi^2_{\mathrm{B}}\rangle+|\psi^1_{\mathrm{A}}\rangle\otimes(|\Delta^{11}_{\mathrm{B}}\phi^1_{\mathrm{B}}\rangle+|\Delta^{12}_{\mathrm{B}}\phi^2_{\mathrm{B}}\rangle)\\
&\qquad+|\psi^2_{\mathrm{A}}\rangle\otimes(|\Delta^{21}_{\mathrm{B}}\phi^1_{\mathrm{B}}\rangle+|\Delta^{22}_{\mathrm{B}}\phi^2_{\mathrm{B}}\rangle)).
\end{aligned}
$$

In virtue of the constraints, we get that

$$
\begin{aligned}
&H_{\mathrm{AB}}(t)\cdot\frac{1}{2}(\boldsymbol{\Psi}^1_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^2_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^3_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^4_{\mathrm{AB}}(t))=\\
&\quad\frac{1}{2}\big[(\lambda''|\psi^1_{\mathrm{A}}\rangle+\lambda'''|\psi^2_{\mathrm{A}}\rangle)\otimes|\phi^1_{\mathrm{B}}\rangle\\
&\qquad+(|\Delta^{12}_{\mathrm{A}}\psi^1_{\mathrm{A}}\rangle+|\Delta^{22}_{\mathrm{A}}\psi^2_{\mathrm{A}}\rangle)\otimes(|\phi^1_{\mathrm{B}}\rangle+|\phi^2_{\mathrm{B}}\rangle)\\
&\qquad+(|\psi^1_{\mathrm{A}}\rangle+|\psi^2_{\mathrm{A}}\rangle)\otimes(|\Delta^{11}_{\mathrm{B}}\phi^1_{\mathrm{B}}\rangle+|\Delta^{12}_{\mathrm{B}}\phi^2_{\mathrm{B}}\rangle)\big].
\end{aligned}
$$

Such a state does not contain any state bi-orthogonal to $\frac{1}{2}(\boldsymbol{\Psi}^1_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^2_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^3_{\mathrm{AB}}(t)+\boldsymbol{\Psi}^4_{\mathrm{AB}}(t))$ only if $\lambda''|\psi^1_{\mathrm{A}}\rangle+\lambda'''|\psi^2_{\mathrm{A}}\rangle=\lambda''''(|\psi^1_{\mathrm{A}}\rangle+|\psi^2_{\mathrm{A}}\rangle)$, which imposes that $\lambda''=\lambda'''=\lambda''''$. We can repeat this proof with the indices ii' for the system A and $1j$ for the system B instead of 12, as it was the case in the previous proof, and we obtain that $|\Delta^{ij}_{\mathrm{B}}\phi^j_{\mathrm{B}}\rangle=|\Delta^{i'j}_{\mathrm{B}}\phi^j_{\mathrm{B}}\rangle=|\Delta^j_{\mathrm{B}}\phi^j_{\mathrm{B}}\rangle$, and $|\Delta^{ij}_{\mathrm{A}}\psi^i_{\mathrm{A}}\rangle=|\Delta^{i1}_{\mathrm{A}}\psi^i_{\mathrm{A}}\rangle-\lambda(j)|\psi^i_{\mathrm{A}}\rangle=|\Delta^i_{\mathrm{A}}\psi^i_{\mathrm{A}}\rangle-\lambda(j)|\psi^i_{\mathrm{A}}\rangle$ (where $|\Delta^i_{\mathrm{A}}\psi^i_{\mathrm{A}}\rangle$ does not depend on j while $\lambda(j)$ and $|\Delta^j_{\mathrm{B}}\phi^j_{\mathrm{B}}\rangle$ do not depend on i). Therefore

$$
\begin{aligned}
&H_{\mathrm{AB}}(t)\cdot|\psi^i_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle=\\
&|\Delta^i_{\mathrm{A}}\psi^i_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle+|\psi^i_{\mathrm{A}}\rangle\otimes|\Delta^j_{\mathrm{B}}\phi^j_{\mathrm{B}}\rangle-\lambda(j)|\psi^i_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle,
\end{aligned}
$$

which fulfills the (2) provided we proceed to the following identifications: $H_{\mathrm{A}}(t)\cdot|\psi^i_{\mathrm{A}}\rangle=|\Delta^i_{\mathrm{A}}\psi^i_{\mathrm{A}}\rangle$ and $H_{\mathrm{B}}(t)\cdot|\phi^j_{\mathrm{B}}\rangle=|\Delta^j_{\mathrm{B}}\phi^j_{\mathrm{B}}\rangle-\lambda(j)|\phi^j_{\mathrm{B}}\rangle$. This ends the proof of the necessary condition of the main theorem.

B) Proof of the sufficient condition. Let us assume that the full Hamiltonian can be factorised according to (2). Let us consider the product state $\psi^{\mathrm{red}}_{\mathrm{A}}(t')\otimes\psi^{\mathrm{red}}_{\mathrm{B}}(t')$, where $\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t')$ is a solution of the reduced Schrödinger equation $i\hbar\partial_{t'}\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t')=H_{\mathrm{A(B)}}(t')\cdot\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t')$ for the initial condition $\psi^{\mathrm{red}}_{\mathrm{A(B)}}(t)=\psi_{\mathrm{A(B)}}(t)$. Obviously, $i\hbar\partial_{t'}\psi^{\mathrm{red}}_{\mathrm{A}}(t')\otimes\psi^{\mathrm{red}}_{\mathrm{B}}(t')=H_{\mathrm{AB}}(t')\psi^{\mathrm{red}}_{\mathrm{A}}(t')\otimes\psi^{\mathrm{red}}_{\mathrm{B}}(t')$ and $\boldsymbol{\Psi}_{\mathrm{AB}}(t)=\psi^{\mathrm{red}}_{\mathrm{A}}(t)\otimes\psi^{\mathrm{red}}_{\mathrm{B}}(t)$ so that, as the solution of Schrödinger is univoquely determined by the initial conditions at time t, $\boldsymbol{\Psi}_{\mathrm{AB}}(t')=\psi^{\mathrm{red}}_{\mathrm{A}}(t')\otimes\psi^{\mathrm{red}}_{\mathrm{B}}(t')$, $\forall t'\geq t$, which ends the proof of the main theorem.

Some Remarks and Comments

The proof of the sufficient condition is also valid for infinitely dimensional Hilbert spaces. We expect that the necessary condition is also valid in infinitely dimensional Hilbert spaces provided the Hamiltonian is sufficiently regular, but this is presently a mere conjecture.

Actually, many results that are presented in the present section already appeared in [8] two years ago. After the completion of that work, we were kindly informed that very similar results were obtained independently by Gemmer and Mahler [9]. In this work, the authors showed that if two quantum systems mutually interact and the degree of entanglement remains constant in time for all pure states (not only factorisable pure states but also entangled ones), the Hamiltonian necessarily factorises into the sum of individual Hamiltonians. Our main theorem is slightly more general in the sense that it shows that the same necessary condition can be deduced from the weaker assumption that all pure factorisable states remain factorisable during their temporal evolution. The authors also proved that particular pure factorisable states remain factorisable during the evolution if and only if (5) is fulfilled, so to say if and only if the the Hamiltonian factorises into the sum of individual effective Hamiltonians. Although the final results are very close to each other, both approaches are quite different. For instance, the geometrical properties of the bi-orthogonal decomposition [10] (see appendix) and of the bi-orthogonality play a crucial role in our proofs, while this is not true for what concerns the proofs presented in [9] (based on a pseudo-Schrödinger-equation) which are more algebrical and less geometrical than ours. Because of this, our proofs are simpler and more intuitive. The price to pay, nevertheless, is that our approach remains confined to the situation in which states are factorisable. Thanks to their more sophisticated mathematical treatment the authors of the [9] managed to derive an expression aimed at quantifying the amount of entanglement that occurs during the interaction of two quantum systems that do not remain factorisable. Remark that the main theorem and the estimation of the rate of generation of entanglement given in the appendix are also easy to prove on the basis of the results obtained by Cirac et al. [11] on the entangling power of non-local

Hamiltonians, but in the case of two interacting qubits only.

Finally, it is worth noting that the condition (5) encountered in the proof of the sufficient condition of the lemma can be generalised to factorisable non-necessarily pure states. This is the essence of the following theorem that was proven in [8] and that we reproduce without proof (the proof is straightforward).

Theorem 2

If initially a bipartite system is prepared in a factorisable (non-necessarily pure) state: $\rho_{\mathrm{AB}}(t=0) = \rho_{\mathrm{A}}(t=0) \otimes \rho_{\mathrm{B}}(t=0)$, and $\forall t \geq 0$,

$$\begin{aligned} H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t) = {} & (H_{\mathrm{A}}^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{A}}(t)) \otimes \rho_{\mathrm{B}}(t) \\ & + \rho_{\mathrm{A}}(t) \otimes (H_{\mathrm{B}}^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{B}}(t)), \end{aligned} \tag{10}$$

where

$$H_{\mathrm{A}}^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{B}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t))$$

and

$$\begin{aligned} H_{\mathrm{B}}^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{B}}(t) = {} & \mathrm{Tr}_{\mathrm{A}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t)) \\ & - (\mathrm{Tr}_{\mathrm{AB}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t))) \cdot \rho_{\mathrm{B}}(t), \end{aligned}$$

then, necessarily, the state remains factorisable during the interaction: $\rho_{\mathrm{AB}}(t) = \rho_{\mathrm{A}}(t) \otimes \rho_{\mathrm{B}}(t)\ \forall t \geq 0$.

In this approach, and with this definition of effective Hamiltonians, we face the following problem: it is easy to show that the sufficient condition expressed by (10) is also necessary in the case of pure states (because then (5) must be valid at any time, in virtue of the necessary condition of the lemma but the condition (5) implies the condition (10) in virtue of the Schrödinger equation (1). Now, in the case of non-pure states, the sufficient condition expressed by (10) is in general not necessary, as shows the following counterexample. If initially, the bipartite system is prepared in a factorisable state: $\rho_{\mathrm{AB}}(t=0) = \rho_{\mathrm{A}}(t=0) \otimes \rho_{\mathrm{B}}(t=0)$, and that $\forall t \geq 0, H_{\mathrm{AB}}(t) = \rho_{\mathrm{AB}}(t=0)$, then it is easy to check that $\rho_{\mathrm{AB}}(t=0) = \rho_{\mathrm{AB}}(t) \forall t \geq 0$, $H_{\mathrm{A}}^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{B}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t)) = \rho_{\mathrm{A}}^2(t=0) \cdot \mathrm{Tr}_{\mathrm{B}}\rho_{\mathrm{B}}^2(t=0)$, $H_{\mathrm{B}}^{\mathrm{eff.}}(t) \cdot \rho_{\mathrm{B}}(t) = \mathrm{Tr}_{\mathrm{A}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t)) - (\mathrm{Tr}_{\mathrm{AB}}(H_{\mathrm{AB}}(t)\rho_{\mathrm{AB}}(t))) \cdot \rho_{\mathrm{B}}(t) = \mathrm{Tr}_{\mathrm{A}}\rho_{\mathrm{A}}^2(t=0) \cdot \rho_{\mathrm{B}}^2(t=0) - \mathrm{Tr}_{\mathrm{A}}\rho_{\mathrm{A}}^2(t=0) \cdot \mathrm{Tr}_{\mathrm{B}}\rho_{\mathrm{B}}^2(t=0) \cdot \rho_{\mathrm{B}}(t=0)$ and it is easy to check that in general (10) is not valid when the initial state is not pure, so to say when it is not a product of pure states. This led us recently to redefine the effective Hamiltonians in order to be able to treat also the case of non-pure states. These results are encapsulated in the following theorem:

Theorem 3

A bipartite system initially prepared in a factorisable (non-necessarily pure) state $(\rho_{\mathrm{AB}}(t=0) = \rho_{\mathrm{A}}(t=0) \otimes \rho_{\mathrm{B}}(t=0))$ remains in a factorisable state throughout the evolution $(\rho_{\mathrm{AB}}(t) = \rho_{\mathrm{A}}(t) \otimes \rho_{\mathrm{B}}(t) \forall t \geq 0)$ if and only if the effect of the Hamiltonian can be factorised as follows: $\forall t \geq 0$

$$\begin{aligned} [H_{\mathrm{AB}}(t), \rho_{\mathrm{AB}}(t)] = {} & ([H_{\mathrm{A}}^{\mathrm{eff}'\cdot}(t), \rho_{\mathrm{A}}(t)]) \otimes \rho_{\mathrm{B}}(t) \\ & + \rho_{\mathrm{A}}(t) \otimes ([H_{\mathrm{B}}^{\mathrm{eff}'\cdot}(t), \rho_{\mathrm{B}}(t)]), \end{aligned} \tag{11}$$

where

$$[H_{\mathrm{A}}^{\mathrm{eff}'\cdot}(t), \rho_{\mathrm{A}}(t)] = \mathrm{Tr}_{\mathrm{B}}([H_{\mathrm{AB}}(t), \rho_{\mathrm{AB}}(t)])$$

and

$$\begin{aligned} [H_{\mathrm{B}}^{\mathrm{eff}'\cdot}(t) \cdot \rho_{\mathrm{B}}(t)] = {} & \mathrm{Tr}_{\mathrm{A}}([H_{\mathrm{AB}}(t), \rho_{\mathrm{AB}}(t)]) \\ & - (\mathrm{Tr}_{\mathrm{AB}}([H_{\mathrm{AB}}(t), \rho_{\mathrm{AB}}(t))]) \cdot \rho_{\mathrm{B}}(t) \\ = {} & \mathrm{Tr}_{\mathrm{A}}([H_{\mathrm{AB}}(t), \rho_{\mathrm{AB}}(t)]). \end{aligned}$$

It is worth noting that, although their effects are unambiguously defined in terms of the effect of the global Hamiltonian, there does not necessarily exist effective Hamiltonians (self-adjoint operators) $H^{\mathrm{eff}'\cdot}$ that satisfy the previous definitions. Therefore the commutators that appear in these definitions must be considered symbolically. Nevertheless, their trace is equal to zero, as would be the case with real commutators.

Proof of the Theorem 3

When we describe the state of the system by a density matrix, its evolution obeys the von Neumann equation

$$i\hbar\partial_t \rho_{\mathrm{AB}}(t) = [H_{\mathrm{AB}}(t), \rho_{\mathrm{AB}}(t)], \tag{12}$$

where $[X, Y]$ represents the commutator of two operators X and Y. If (11) is satisfied, we have that:

$$\begin{aligned} i\hbar\partial_t \rho_{\mathrm{AB}}(t) = {} & ([H_{\mathrm{A}}^{\mathrm{eff}'\cdot}(t), \rho_{\mathrm{A}}(t)]) \otimes \rho_{\mathrm{B}}(t) \\ & + \rho_{\mathrm{A}}(t) \otimes ([H_{\mathrm{B}}^{\mathrm{eff}'\cdot}(t), \rho_{\mathrm{B}}(t)]) \end{aligned}$$

Let us consider the product state $\rho_{\mathrm{A}}^{\mathrm{red}}(t) \otimes \rho_{\mathrm{B}}^{\mathrm{red}}(t)$, where $\rho_{\mathrm{A(B)}}^{\mathrm{red}}(t)$ is a solution of the reduced von Neumann equation $i\hbar\partial_t \rho_{\mathrm{A(B)}}^{\mathrm{red}}(t) = [H_{\mathrm{A(B)}}^{\mathrm{eff}'\cdot}(t), \rho_{\mathrm{A(B)}}^{\mathrm{red}}(t)]$ for the initial condition $\rho^{\mathrm{red}}_{\mathrm{A(B)}}(t=0)=\rho_{\mathrm{A(B)}}(t=0)$. In virtue of the Leibniz rule and of the condition (11), we get that $i\hbar\partial_t(\rho^{\mathrm{red}}_{\mathrm{A}}(t)\otimes\rho^{\mathrm{red}}_{\mathrm{B}}(t))=[H_{\mathrm{AB}}(t),\rho^{\mathrm{red}}_{\mathrm{A}}(t)\otimes\rho^{\mathrm{red}}_{\mathrm{B}}(t)]$ and $\rho_{\mathrm{AB}}(t=0)=\rho^{\mathrm{red}}_{\mathrm{A}}(t=0)\otimes\rho^{\mathrm{red}}_{\mathrm{B}}(t=0)$ so that, as (12) is deterministic, $\rho_{\mathrm{AB}}(t)=\rho^{\mathrm{red}}_{\mathrm{A}}(t)\otimes\rho^{\mathrm{red}}_{\mathrm{B}}(t)$, $\forall t\geq 0$.

Conversely, if $\rho_{\mathrm{AB}}(t)=\rho^{\mathrm{red}}_{\mathrm{A}}(t)\otimes\rho^{\mathrm{red}}_{\mathrm{B}}(t)\forall t\geq 0$, then, as the von Neumann evolution (12) is trace preserving, we can, without loss of generality, assume that $\mathrm{Tr}\rho^{\mathrm{red}}_{\mathrm{A}}(t)=\mathrm{Tr}\rho^{\mathrm{red}}_{\mathrm{B}}(t)=1$, $\forall t\geq 0$ so that, in virtue of the properties of the trace, $\mathrm{Tr}_{\mathrm{A(B)}}(\partial_t\rho_{\mathrm{AB}})=(\partial_t\rho^{\mathrm{red}}_{B(A)})$, which ends the proof.

Note that (5), (10) and (11) are linear in the coupling Hamiltonian H_{AB} and are automatically satisfied when (2) is satisfied. Nevertheless it is non-linear in ρ_{AB}. Moreover, the effective potential that acts onto say the A particle is likely to depend on the state of the B particle, a situation that does not occur if we impose that all product states remain product states.

3. The Factorisation Approximation and the Classical Limit

3.1. The Decoherence Program, the PS Criterion and the Classical Limit

It is worth noting that the proof of the lemma, and of the theorems 2 and 3 as well, are also valid when the systems A and B are infinite dimensional, for instance when they are localised particles that interact through a central potential. In this section we shall consider only this very simple situation and apply the results of the lemma at the light of the predictability sieve criterion introduced by Zurek in the framework of the decoherence approach [5–7]. This program is an attempt to solve fundamental paradoxes of quantum mechanics (the apparent temporal irreversibility of the measurement process, the measurement problem that deals with the separation quantum-classical, and so on). Two ingredients are essential in this approach:

- decoherence is seen as an aspect of entanglement (this property is trivial if we measure the coherence of a system by the Shannon-von Neumann information of its reduced density matrix, obtained after tracing out the rest of the world (environment)).
- the measurement process is seen as the interaction between a quantum system (that could even include the supposedly quantum measuring apparatus) and a complex (supposedly quantum) environment, typically an infinite bath of oscillators [5–7].

Traditionnally, such (open) quantum systems are described by semi-phenomenological irreversible equations, which predict the occurence of an (irreversible) increase of entanglement between the observed system and the environment. Decoherence is then directly related to this entanglement increase, via the well-known properties of Shannon-von Neumann information. Zurek postulated that, roughly speaking, during the evolution, our brain selected, during the interactive process of creation of a world view, the classical islands that correspond to the minimal increase of Shannon-von Neumann entropy [6]. This is called the EINselection (EIN for environment induced), and this procedure has been referred to as the predictability sieve criterion (see the updated version of [5] for a review). The emergence of a classical world that obeys EINselections can be explained following two ways:

A) they correspond to maximal (Shannon-von Neumann) information; it is well plausible that our brain selects the features of the natural world that contain maximal information;

B) we can also invoke an argument of structural stability: superposition of states that would belong to such islands would be destroyed very quickly by the decoherence process which radiates irremediably the coherence (or information or Shannon-von Neumann negentropy) into the environment [5].

Up to now, the predictability sieve criterion was only (to the knowledge of the author) applied to open quantum systems, so to say, it was assumed that the environment is complex, with a Poincaré recurrence (or revival) time tending to infinity FAPP. The decoherence process itself was studied in mesoscopic situations, theoretically and experimentally as well, [12], but it was usually assumed that the real, macroscopic situation corresponded to the limit of infinitely small (FAPP) decoherence times. In this section, we shall apply the predictability sieve criterion to a very simple situation during which the system A and the environment B are two distinguishable particles and are described by a (pure) scalar wave function that obeys the non-relativistic Schrödinger equation. We shall also assume that their interaction potential is an action at distance that is time-independent and invariant under spatial translations (a Coulombian interaction for instance), this is a standard text-book situation that was deeply studied, for instance in the framework of scattering theory. The systems A and B fulfill thus (in the non-relativistic regime) the following Schrödinger equation:

$$i\hbar\partial_t\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}},t) = -\Big(\frac{\hbar^2}{2m_{\mathrm{A}}}\Delta_{\mathrm{A}} + \frac{\hbar^2}{2m_{\mathrm{B}}}\Delta_{\mathrm{B}}\Big)\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}},t) + V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}})\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}},t), \qquad (13)$$

where $\Delta_{\mathrm{A(B)}}$ is the Laplacian operator in the A(B) coordinates.

3.2. *PS Criterion for two Interacting Particles: the Effective Field Regime*

Let us now consider that the system *A* is the quantum system that interests us, and that the other system is its environment. Actually the argument is symmetrical, as we shall see, so that this choice is a mere convention. In order to identify the classical islands, according to the PS criterion, we must identify the states that exhibit maximal coherence or maximal Shannon-von Neumann information. We assume here that the full state is pure. Then, the classical islands correspond to the states that initially, and during their interaction as well, remain factorisable (more precisely in a pure factorisable state) so that, in virtue of the lemma, the Hamiltonian may not couple the state to a biorthogonal state and (5) is fulfilled. This equation correponds to what is sometimes called in the literature the mean field or effective field approximation. It expresses that everything happens as if each particle (A(B)) ”felt” the influence of the other particle as if it was diluted with a probability distribution equal to the quantum value $|\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{B(A)}}|^2$. It corresponds also to the concept of a droplet or diluted particle. It can be shown [8] that, in the static case, the condition [13] reduces to the so-called Hartree approximation [13]. Let us consider a bound state of the Helium atom for instance, and let us neglect the fermionic exchange contributions, the spins of the electrons and of the nucleus and so on. The time independent (electronic) Schrödinger equation is then the following:

$$\begin{aligned} &E_{\mathrm{AB}}\cdot\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}}) \\ &= \Big(-\frac{\hbar^2}{2m_{\mathrm{A}}}\Delta_{\mathrm{A}} + V_{\mathrm{A}} - \frac{\hbar^2}{2m_{\mathrm{B}}}\Delta_{\mathrm{B}} + V_{\mathrm{B}}\Big)\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}}) \qquad (14)\\ &\quad + V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}})\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}}),\end{aligned}$$

where V_{A} and V_{B} represent the external fields (for instance the Coulombian nuclear field), while V_{AB} represents the Coulombian repulsion between the electrons A and B. Let us assume that this equation admits a factorisable solution $\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}}) = \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}})\cdot\psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}})$; then, as is shown in [8], we can derive, up to elementary manipulations, the following consistency condition:

$$\begin{aligned} &\Big(E_{\mathrm{AB}} - \Big\langle -\frac{\hbar^2}{2m_{\mathrm{A}}}\Delta_{\mathrm{A}} + V_{\mathrm{A}}\Big\rangle_{\mathrm{A}} \\ &\quad -\Big\langle -\frac{\hbar^2}{2m_{\mathrm{A}}}\Delta_{\mathrm{B}} + V_{\mathrm{B}}\Big\rangle_{\mathrm{B}}\Big)\cdot\psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}})\cdot\psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}) \qquad (15)\\ &= (\langle V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}})\rangle_{\mathrm{A}} + \langle V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}})\rangle_{\mathrm{B}} \\ &\quad - V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}}))\cdot\psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}})\cdot\psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}).\end{aligned}$$

Equivalently, when the wave-function does not vanish, the following condition must be satisfied:

$$\begin{aligned} V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}}) &= \langle V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}})\rangle_{\mathrm{A}} + \langle V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}})\rangle_{\mathrm{B}} \\ &\quad - \langle V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}}-\boldsymbol{r}_{\mathrm{B}})\rangle_{\mathrm{AB}}. \qquad (16)\end{aligned}$$

This is nothing else than the condition (5) in a static form. Obviously, in this regime particles behave as if they were discernable, and constituted of a dilute, continuous medium distributed in space according to the quantum distribution $|\psi_{\mathrm{A(B)}}|^2(r_{\mathrm{A(B)}},t)$.

3.3. *Special case 1: the Test-particle Regime*

As the potential does depend on the relative position $\boldsymbol{r}_{\mathrm{rel}} = \boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{B}}$ only, it is convenient to pass to the center of mass coordinates:

$$\begin{aligned} &i\hbar\partial_t\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{CM}},\boldsymbol{r}_{\mathrm{rel}},t) \\ &= -\Big(\frac{\hbar^2}{2(m_{\mathrm{A}}+m_{\mathrm{B}})}\Delta_{\mathrm{CM}} + \frac{\hbar^2}{2\mu}\Delta_{\mathrm{rel}}\Big)\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{CM}},\boldsymbol{r}_{\mathrm{rel}},t) \qquad (17)\\ &\quad + V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{rel}})\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{CM}},\boldsymbol{r}_{\mathrm{rel}},t),\end{aligned}$$

where $\boldsymbol{r}_{\mathrm{CM}} = (m_{\mathrm{A}}\boldsymbol{r}_{\mathrm{A}} + m_{\mathrm{B}}\boldsymbol{r}_{\mathrm{B}})/(m_{\mathrm{A}}+m_{\mathrm{B}})$ and $\mu = m_{\mathrm{A}}m_{\mathrm{B}}/(m_{\mathrm{A}}+m_{\mathrm{B}})$. As is well-known, the previous equation is then separable, which means that if, initially, the wave-function is factorisable in these coordinates, it will remain so during the evolution. Now, we are interested in situations for which the wave-function is initially factorisable according to the partition of the Hilbert space that is induced by the systems A and B. In general, such a wave-function is not factorisable in the coordinates of the center of mass. Formally, if $\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}},\boldsymbol{r}_{\mathrm{B}},t=0) = \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}},t=0)\cdot\psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}},t=0)$, $\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{CM}},\boldsymbol{r}_{\mathrm{rel}},t=0) = \int \mathrm{d}\omega A(\omega)\psi^{\omega}_{\mathrm{CM}}(\boldsymbol{r}_{\mathrm{CM}},t=0)\cdot\psi^{\omega}_{\mathrm{rel}}(\boldsymbol{r}_{\mathrm{rel}},t=0)$, where $\mathrm{A}(\omega)$ is a generally non-peaked amplitude distribution. Then, at time t,

$\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{CM}}, \boldsymbol{r}_{\mathrm{rel}}, t) = \int \mathrm{d}\omega A(\omega) \psi^{\omega}_{\mathrm{CM}}(\boldsymbol{r}_{\mathrm{CM}}, t) \cdot \psi^{\omega}_{\mathrm{rel}}(\boldsymbol{r}_{\mathrm{rel}}, t)$, where $\psi^{\omega}_{\mathrm{CM}}(\boldsymbol{r}_{\mathrm{CM}}, t)$ obeyed during the time interval $[0, t]$ a free Schrödinger evolution for the initial condition $\psi^{\omega}_{\mathrm{CM}}(\boldsymbol{r}_{\mathrm{CM}}, t = 0)$ while $\psi^{\omega}_{\mathrm{rel}}(\boldsymbol{r}_{\mathrm{rel}}, t)$ was submitted to the interaction potential $V_{\mathrm{AB}}(r_{\mathrm{rel}})$. In general, $\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}}, \boldsymbol{r}_{\mathrm{B}}, t)$ is no longer factorisable into a product of the form $\psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t) \cdot \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t)$. Actually, this is not astonishing because, in virtue of Noether's theorem the full momentum is conserved during the evolution. Therefore the recoil of one of the two particles could be used in order to determine (up to the initial undeterminacy of the centre of mass) what is the recoil of the second particle. The existence of such correlations is expressed by the entanglement of the full wave-function. On the basis of such general considerations we expect that entanglement is very likely to occur due to the interaction between the two particles.

Nevertheless, if $m_{\mathrm{A}} \ll m_{\mathrm{B}}$, that the initial state is factorisable and that the B particle is initially at rest and well localized, it can be shown that the Hamiltonian does not couple the state to a biorthogonal state and (5) is fulfilled. Indeed, if we let coincide the origin of the system of coordinates associated to the particle B with its location, and that we neglect its recoil as well as its dispersion, the following approximations are valid: $\boldsymbol{r}_{\mathrm{CM}} \approx \boldsymbol{r}_{\mathrm{B}} \approx 0$, $\boldsymbol{r}_{\mathrm{rel}} \approx \boldsymbol{r}_{\mathrm{A}} - 0 = \boldsymbol{r}_{\mathrm{A}}$, $\psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t) \approx \psi_{\mathrm{rel}}(\boldsymbol{r}_{\mathrm{rel}}, t)$ and $\psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t) \approx \psi_{\mathrm{CM}}(\boldsymbol{r}_{\mathrm{CM}}, t)$. Moreover, $\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}}, \boldsymbol{r}_{\mathrm{B}}, t = 0) = \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t = 0) \cdot \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t = 0) \approx \psi_{\mathrm{rel}}(\boldsymbol{r}_{\mathrm{rel}}, t = 0) \cdot \psi_{\mathrm{CM}}(\boldsymbol{r}_{\mathrm{CM}}, t = 0) \approx \boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{CM}}, \boldsymbol{r}_{\mathrm{rel}}, t = 0)$. At time t, $\boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{CM}}, \boldsymbol{r}_{\mathrm{rel}}, t) \approx \psi_{\mathrm{rel}}(\boldsymbol{r}_{\mathrm{rel}}, t = 0) \cdot \psi_{\mathrm{CM}}(\boldsymbol{r}_{\mathrm{CM}}, t) \approx \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t) \cdot \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t) \approx \boldsymbol{\Psi}(\boldsymbol{r}_{\mathrm{A}}, \boldsymbol{r}_{\mathrm{B}}, t)$. The separability of the full system into its components A and B is thus ensured, in good approximation, during the scattering process and for bound states as well.

3.4. Special case 2: The Material Point Regime

Another situation that is of physical interest is the situation of mutual scattering of two well localized wave packets whenever we can neglect the quantum extension of the interacting particles. This will occur when the interaction potential V_{AB} is smooth enough and the particles A and B are described by wave packets the extension of which is small in comparison to the typical lenght of variation of the potential. It is well known that in this regime, when the de Broglie wave lenghts of the wave packets are small enough, it is consistent to approximate quantum wave mechanics by its geometrical limit, which is classical mechanics. For instance the quantum differential cross sections converge in the limit of small wave-lenghts to the corresponding classical cross sections. Ehrenfest's theorem also predicts that, when we can neglect the quantum fluctuations, which is the case here, the average motions are nearly classical and provide a good approximation to the behaviour of the full wave-packet in so far we consider it as a material point. In this regime we can in good approximation replace the interaction potential by the first order term of its Taylor development around the centers of the wave-packets associated to the particles A and B:

$$\begin{aligned}
V_{\mathrm{AB}}(\boldsymbol{r}_{\mathrm{A}} - \boldsymbol{r}_{\mathrm{B}}) \approx{}& V_{\mathrm{AB}}(\langle \boldsymbol{r}_{\mathrm{A}} \rangle_t - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t) \\
&+ \nabla_{\mathrm{A}} V_{\mathrm{AB}}(\langle \boldsymbol{r}_{\mathrm{A}} \rangle_t - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t) \cdot (\boldsymbol{r}_{\mathrm{A}} - \langle \boldsymbol{r}_{\mathrm{A}} \rangle_t) \\
&+ \nabla_{\mathrm{B}} V_{\mathrm{AB}}(\langle \boldsymbol{r}_{\mathrm{A}} \rangle - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t) \cdot (\boldsymbol{r}_{\mathrm{B}} - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t).
\end{aligned}$$

Then the evolution equation is in good approximation separable into the coordinates $\boldsymbol{r}_{\mathrm{A}}, \boldsymbol{r}_{\mathrm{B}}$, and we have that, when $\Psi(\boldsymbol{r}_{\mathrm{A}}, \boldsymbol{r}_{\mathrm{B}}, t = 0) = \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t = 0) \cdot \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t = 0)$, then, at the time t, $\Psi(\boldsymbol{r}_{\mathrm{A}}, \boldsymbol{r}_{\mathrm{B}}, t) \approx \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t) \cdot \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t)$, where

$$\begin{aligned}
i\hbar \partial_t \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t) \approx{}& -\frac{\hbar^2}{2m_{\mathrm{A}}} \Delta_{\mathrm{A}} \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t) \\
&+ [V_{\mathrm{AB}}(\langle \boldsymbol{r}_{\mathrm{A}} \rangle_t) - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t) \\
&\quad + \nabla_{\mathrm{A}} V_{\mathrm{AB}}(\langle \boldsymbol{r}_{\mathrm{A}} \rangle_t - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t) \cdot (\boldsymbol{r}_{\mathrm{A}} - \langle \boldsymbol{r}_{\mathrm{A}} \rangle_t)] \psi_{\mathrm{A}}(\boldsymbol{r}_{\mathrm{A}}, t)
\end{aligned} \tag{18}$$

$$\begin{aligned}
i\hbar \partial_t \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t) \approx{}& -\frac{\hbar^2}{2m_{\mathrm{B}}} \Delta_{\mathrm{B}} \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t) \\
&+ [\nabla_{\mathrm{B}} V_{\mathrm{AB}}(\langle \boldsymbol{r}_{\mathrm{A}} \rangle_t - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t) \cdot (\boldsymbol{r}_{\mathrm{B}} - \langle \boldsymbol{r}_{\mathrm{B}} \rangle_t)] \psi_{\mathrm{B}}(\boldsymbol{r}_{\mathrm{B}}, t),
\end{aligned} \tag{19}$$

Note that the Bohmian velocities associated with the particles A and B are factorisable only when the full state is factorisable. Otherwise, the velocity of a particle depends non-locally on the location of both particles.

In summary, we see thus that, in the simple case considered in this section, the classical islands EINselected in virtue of the PS criterion are regions of the Hilbert space where the mean or effective field approximation (or Hartree approximation in the static case) is valid. Then, the interaction factorises into the sum of two effective potentials that act separately on both particles and express the average influence due to the presence of the other particle (which is not true in general and certainly not inside the atom).

In particular, in the test-particle limit, the effective potential undergone by the massive particle is close to zero, and when the heavy particle is well localised, its average, effective, potential is close to the real potential undergone by the light "test-particle". In the classical limit (material points), the quantum internal structure of the interacting particles can be neglected and the potential is equivalent to the sum of the effective potentials in good approximation.

Finally, it is worth noting that in general the superposition principle is not valid inside classical islands: the dynamical constraints (5, 10, 11) being non-linear even in the static case (16), the superposition of two states that belong to classical islands does not in general belong to such an island. Another way to formulate this remark is to say that decoherence-free subspaces are most often one-dimensional. Moreover, when the full state is a product of non-pure states, it can happen that, although it remains factorisable in time and although the system is closed and undergoes a unitary evolution, which preserves the Shannon-von Neumann entropy of the full state, the reduced evolutions do no longer preserve the entropies of the reduced states. In such a situation, the forementioned symmetry between system and environment is broken. This situation is potentially richer because it allows transfers of entropy between the system and the environment but it is out of the scope of the present paper to study all the possibilities that appear in this case.

4. Conclusions and Comments

Originally, the present work was motivated by the results presented in [14, 15]. In these papers it is argued and shown that retrievable, usable quantum information can be transferred in a scheme which, in striking contrast to the quantum teleportation schemes, requires no external channel and does not involve the transfer of a quantum state from one subsystem to the other. Although other specific quantum ingredients are present (such as entanglement), entanglement-free interaction between two mutually scattering particles (in the three dimensional, physical space) plays a crucial role in this scheme. The previous remarks suggest that localisation of at least one of the particles is one of the necessary ingredients of such protocols for quantum information transfer. For instance, in the test-particle limit the massive particle is localised while in the classical limit, both particles are localised. It is easy to show that, if at least one of the two interacting particles is not well localised (bilocated for instance) and the particles interact through a position-dependent potential (action at a distance), they are highly likely to end up in an entangled state.

A conclusion of the first part of this work (main theorem) could be: in quantum mechanics to interact means nearly always to entangle. We showed that real interactions do necessarily generate entanglement (the inverse result, that it is impossible to generate entanglement without turning on an interaction, is rather trivial).

Considered so, the degree of entanglement of large (generic) classes of states for large (generic) classes of systems ought to increase with time, which would indicate some analogy between entanglement and entropy. Note, however, that the temporal reversibility of the Schrödinger equation implies that the degree of entanglement could also decrease with time so that we face a paradox analog to the famous Loschmidt paradox which emphasises the apparent contradiction between the temporal asymmetry of the second principle of thermodynamics and the temporal symmetry of fundamental interactions. Actually, it seems that nowadays the Shannon-von Neumann entropy plays more and more the central (and still mysterious) role played by the Boltzmann entropy in statistical mechanics and thermodynamics. In quantum information, for instance, it is often implicitly assumed that the corresponding negentropy contents the "reality" of the system ("all" is (quantum) information), and the decoherence approach confers to (quantum) information, through the PScriterion, a supraphysical role: the organisation of our brain (the way we think) as well as the so-called measurement problem could be ultimately explained in terms of (quantum) information! Recently, the role played by entanglement during phase-transitions was recognised [16], and it could be that we shall be able soon to solve old paradoxes such as the Loschmidt paradox in the new framework of quantum information. Obviously, such considerations are out of the scope of this paper, and we invite the interested reader to consult the reference [17] and references therein.

To the knowledge of the author, nobody else attempted up to now to apply the predictability sieve criterion introduced by Zurek in the very simple and standard situation considered here (the system and the environment as well are described by a pure, scalar wave function and their evolution obeys a classical Hamiltonian). It is gratifying to note that according to our analysis presented in the second section all the regimes that belong to the no-entanglement regime, which is also the classical regime according to the PS criterion, correspond to classical preconceptions about the objects that physicists call particles.[2] The most elaborated model (the droplet or diluted medium model) appeared relatively late in the history (it corresponds to the classical models of the electron developed by Langevin, Poincaré, Abraham and others at the beginning of the last century). In the limit where we can neglect the internal structure of the droplet, we recover as special cases the test-particle limit (the internal structure of the massive particle can be neglected because it is insensitive to the back action of the other one) and the material point regime (we can neglect the quantum extension of both particles). The concept of test particle emerged in the 19th century, in the framework of electrodynamics, and was also useful in general relativity. The material point regime corresponds to the Ehrenfest quasi-classical equation and also to the geometrical limit of quantum wave mechanics which is, as is well-known, Hamiltonian mechanics. The associated concept of material points corresponds to the Galilean and Newtonian paradigms (particles are like little stones of negligible extent that move in empty space), a concept that can be traced back to Democritus. It is difficult to find radically different paradigms in classical physics in order to describe (modellize) the concept of particle.

Note that in all the cases, the systems are separated only in first approximation, and in general entanglement will accompany interaction, in virtue of the main theorem.

One may agree or disagree with our oversimplified choice of what is the system and what is the environment, but our analysis shows that even when it is applied in simple situations, the PS criterion provides unexpected but very natural and useful (ftnote 2) insights: classical islands correspond to the classical preconceptions and exhibit a certain structural stability in time (ftnote 3). It confirms the deep intuition of Schrödinger, already mentioned in the introduction, who described entanglement as the characteristic trait of quantum mechanics, “the one that enforces its entire departure from classical lines of thought” [1].

Appendix

Generation of Entanglement During the Evolution

Let us consider that at the time t the system is prepared in a product state $\boldsymbol{\Psi}_{\mathrm{AB}}(t)=\psi_{\mathrm{A}}(t)\otimes\psi_{\mathrm{B}}(t)$, and let us choose a basis of product states $|\psi^i_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle$ ($i:1\ldots \mathrm{d}_{\mathrm{A}}; j:1\ldots \mathrm{d}_{\mathrm{B}}$, and $\langle\psi^i_{\mathrm{A}}|\psi^j_{\mathrm{A}}\rangle=\delta_{ij}=\langle\phi^i_{\mathrm{B}}|\phi^j_{\mathrm{B}}\rangle$) such that $\psi_{\mathrm{A}}(t)=|\psi^1_{\mathrm{A}}\rangle$ and $\phi_{\mathrm{B}}(t)=|\phi^1_{\mathrm{A}}\rangle$. Then, after a short time δt,

$$\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)=(I+\frac{\delta t}{i\hbar}\cdot H_{\mathrm{AB}}(t))\cdot\boldsymbol{\Psi}_{\mathrm{AB}}(t)+\tau(\delta t^2),$$

where by definition $\tau(\varepsilon^m)$ decreases at least as fast as the mth power of ε when ε goes to zero. In a matricial form, the previous equation becomes

$$\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)=|\psi^1_{\mathrm{A}}\rangle\otimes|\phi^1_{\mathrm{B}}\rangle \tag{20}$$

$$+\frac{\delta t}{i\hbar}\Sigma_{i:1\ldots\mathrm{d}_{\mathrm{A}}}; j:1\ldots\mathrm{d}_{\mathrm{B}}H_{i1j1}|\psi^i_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle+\tau(\delta t^2)$$

where

$$H_{ikjl}=\langle\psi^i_{\mathrm{A}}|\otimes\langle\phi^j_{\mathrm{B}}|H_{\mathrm{AB}}(t)|\psi^k_{\mathrm{A}}\rangle\otimes|\phi^l_{\mathrm{B}}\rangle \tag{21}$$

Equivalently

$$\begin{aligned}\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)&=|\psi^1_{\mathrm{A}}\rangle\otimes|\phi^1_{\mathrm{B}}\rangle\\&+\frac{\delta}{i\hbar}(\Sigma_{i:1\ldots\mathrm{d}_{\mathrm{A}}}H_{i111}|\psi^i_{\mathrm{A}}\rangle\otimes|\phi^1_{\mathrm{B}}\rangle\\&+\Sigma_{j:2\ldots\mathrm{d}_{\mathrm{B}}}H_{11j1}|\psi^1_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle\\&+\Sigma_{i:2\ldots\mathrm{d}_{\mathrm{A}}}; j:2\ldots\mathrm{d}_{\mathrm{B}}H_{i1j1}|\psi^i_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle)+\tau(\delta t^2).\end{aligned}$$

All the components of $\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)$ that are bi-orthogonal to $\boldsymbol{\Psi}_{\mathrm{AB}}(t)$ are contained in the last term of the previous equation: $\Sigma_{i:2\ldots\mathrm{d}_{\mathrm{A}};j:2\ldots\mathrm{d}_{\mathrm{B}}}H_{i1j1}|\psi^i_{\mathrm{A}}\rangle\otimes|\phi^j_{\mathrm{B}}\rangle)$,

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[2] Actually, the simple situations considered here also provide right intuitions even when the environment is more complex. In [6, 7], Zurek and coworkers considered that the environment consist of an infinity of oscillators (bath), a very commonly studied case in the theory of open quantum systems. In [6] it was assumed that the quantum system itself is an oscillator, and the autors showed that the classical islands of the system were the coherent states. It is easy to check that, in our approach, if we consider that V_{A} and V_{B} are harmonic, and that the oscillators A and B are coupled, at the rotating wave approximation by the standard coupling $V_{\mathrm{AB}}=a^+b+ab^+$ where $a(b)$ are the destruction operators for the A(B) quanta, the coherent states obey (5). So, the classical (pointer) states for two coupled oscillators are the coherent states, which provides the right intuition when one oscillator is coupled to a bath. Similarly, when as in [7] the system is a system with a discrete energy spectrum coupled to a bath of slow oscillators, the pointer states (which are in this case the energy eigenstates) are derived modulo the effective field or mean field approximation.

up to $\tau(\delta t^2)$. We can rewrite this equation as follows:

$$\begin{aligned}
&\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t) = \\
&\quad (|\psi_{\mathrm{A}}^1\rangle + \frac{\delta t}{i\hbar}\Sigma_{i:1\ldots \mathrm{d_A}} H_{i111}|\psi_{\mathrm{A}}^i\rangle) \otimes (|\phi_{\mathrm{B}}^1\rangle \\
&\quad + \frac{\delta t}{i\hbar}\Sigma_{j:2\ldots \mathrm{d_B}} H_{11j1}|\phi_{\mathrm{B}}^j\rangle) \\
&\quad + \frac{\delta t}{i\hbar}\Sigma_{i:2\ldots \mathrm{d_A};j:2\ldots \mathrm{d_B}} H_{i1j1}|\psi_{\mathrm{A}}^i\rangle \otimes |\phi_{\mathrm{B}}^j\rangle + \tau(\delta t^2).
\end{aligned} \tag{22}$$

In virtue of the necessary condition of the lemma, if the Hamiltonian couples $\boldsymbol{\Psi}_{\mathrm{AB}}(t=0)$ to states that are bi-orthogonal to it, which means that $\Sigma_{i:2\ldots \mathrm{d_A};j:2\ldots \mathrm{d_B}}|H_{i1j1}|^2 \neq 0$, then, the full state does not remain factorisable. Actually, the development of the first order in δt of the bi-orthogonal or Schmidt decomposition [10] of $\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)$ contains more than one product state, which means that $\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)$ is entangled for δt different from zero but small enough. In order to prove it, let us consider the components of $\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)$ that are bi-orthogonal to $\boldsymbol{\Psi}_{\mathrm{AB}}(t)$. In virtue of the theorem of the bi-orthogonal decomposition of Schmidt [10], we can find $\mathrm{d_A}-1$ normalized states $|\tilde{\psi}_{\mathrm{A}}^i\rangle$ of $\boldsymbol{C}^{\mathrm{d_A}}$ mutually orthogonal and orthogonal to $|\psi_{\mathrm{A}}^1\rangle$, and $\mathrm{d_B}-1$ normalized states $|\tilde{\phi}_{\mathrm{B}}^j\rangle$ of $\boldsymbol{C}^{\mathrm{d_B}}$ mutually orthogonal and orthogonal to $|\phi_{\mathrm{B}}^1\rangle$ such that

$$\begin{aligned}
&\frac{\delta t}{i\hbar}\Sigma_{i:2\ldots \mathrm{d_A};j:2\ldots \mathrm{d_B}} H_{i1j1}|\psi_{\mathrm{A}}^i\rangle \otimes |\phi_{\mathrm{B}}^j\rangle \\
&\quad = \Sigma_{i:2\ldots \min(\mathrm{d_A},\mathrm{d_B})}\alpha_i|\tilde{\psi}_{\mathrm{A}}^i\rangle \otimes |\tilde{\phi}_{\mathrm{B}}^i\rangle.
\end{aligned}$$

Let us now define the state $|\tilde{\psi}_{\mathrm{A}}^{1'}\rangle$ of $\boldsymbol{C}^{\mathrm{d_A}}$ as follows: $|\tilde{\psi}_{\mathrm{A}}^{1'}\rangle = \frac{1}{N_1}\cdot(|\psi_{\mathrm{A}}^1\rangle + \frac{\delta t}{i\hbar}\Sigma_{i:1\ldots \mathrm{d_A}} H_{i111}|\psi_{\mathrm{A}}^i\rangle)$, where N_1 is a normalisation factor, and let us replace the orthonormal basis $\{|\psi_{\mathrm{A}}^1\rangle, |\tilde{\psi}_{\mathrm{A}}^2\rangle, |\tilde{\psi}_{\mathrm{A}}^3\rangle, \ldots, |\tilde{\psi}_{\mathrm{A}}^{\mathrm{d_A}}\rangle\}$ of $\boldsymbol{C}^{\mathrm{d_A}}$ by the orthonormal basis $\{|\tilde{\psi}_{\mathrm{A}}^{1'}\rangle, |\tilde{\psi}_{\mathrm{A}}^{2'}\rangle, |\tilde{\psi}_{\mathrm{A}}^{3'}\rangle, \ldots, |\tilde{\psi}_{\mathrm{A}}^{\mathrm{d}'A}\rangle\}$ of $\boldsymbol{C}^{\mathrm{d_A}}$ that we obtain by the Gram-Schmidt orthonormalisation procedure

$$|\tilde{\psi}_{\mathrm{A}}^{2'}\rangle = \frac{1}{N_2}\cdot(|\tilde{\psi}_{\mathrm{A}}^2\rangle - \langle\tilde{\psi}_{\mathrm{A}}^{1'}|\tilde{\psi}_{\mathrm{A}}^2\rangle \cdot |\tilde{\psi}_{\mathrm{A}}^{1'}\rangle),$$

where N_2 is a normalisation factor.

$$|\tilde{\psi}_{\mathrm{A}}^{3'}\rangle = \frac{1}{N_3}\cdot(|\tilde{\psi}_{\mathrm{A}}^3\rangle - \langle\tilde{\psi}_{\mathrm{A}}^{1'}|\tilde{\psi}_{\mathrm{A}}^3\rangle \cdot |\tilde{\psi}_{\mathrm{A}}^{1'}\rangle - \langle\tilde{\psi}_{\mathrm{A}}^{2'}|\tilde{\psi}_{\mathrm{A}}^3\rangle \cdot |\tilde{\psi}_{\mathrm{A}}^{2'}\rangle),$$

where N_3 is a normalisation factor, and so on. It is easy to check that $|\tilde{\psi}_{\mathrm{A}}^{i'}\rangle = |\tilde{\psi}_{\mathrm{A}}^i\rangle + \tau(\delta t)$. Note that this is no longer true when the dimension $\mathrm{d_A}$ is not finite. We can repeat the same operation in order to replace the orthonormal basis $\{|\phi_{\mathrm{B}}^1\rangle, |\tilde{\phi}_{\mathrm{B}}^2\rangle, |\tilde{\phi}_{\mathrm{B}}^3\rangle, \ldots, |\tilde{\phi}_{\mathrm{B}}^{\mathrm{d_B}}\rangle\}$ of $\boldsymbol{C}^{\mathrm{d_B}}$ by the orthonormal basis $\{|\tilde{\phi}_{\mathrm{B}}^{1'}\rangle, |\tilde{\phi}_{\mathrm{B}}^{2'}\rangle, |\tilde{\phi}_{\mathrm{B}}^{3'}\rangle, \ldots, |\tilde{\phi}_{\mathrm{B}_b}^{\mathrm{d}'}\rangle\}$ of $\boldsymbol{C}^{\mathrm{d_B}}$. Then, after substitition in (22), we obtain

$$\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t) = \Sigma_{i:1\ldots \min(\mathrm{d_A},\mathrm{d_B})}\alpha_i|\tilde{\psi}_{\mathrm{A}}^{i'}\rangle \otimes |\tilde{\phi}_{\mathrm{B}}^{i'}\rangle + \tau(\delta t^2),$$

where

$$\begin{aligned}
|\alpha_1|^2 &= 1 + \tau(\delta t^2)\Sigma_{i:2\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i|^2 \\
&= \frac{\delta t^2}{\hbar^2}\Sigma_{i:2\ldots \mathrm{d_A};j:2\ldots \mathrm{d_B}}|H_{i1j1}|^2 + \tau(\delta t^3)
\end{aligned} \tag{23}$$

The previous equation expresses that the development up to the first order in δt of the bi-orthogonal Schmidt decomposition of $\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)$ contains more than one product state. It is well known that then $\boldsymbol{\Psi}_{\mathrm{AB}}(t+\delta t)$ is an entangled state. Nevertheless, for those who are not familiar with this property, we shall prove directly the result by estimating the linear entropy of the reduced density matrix. By definition, the reduced density matrix ρ_{A} of the system A is equal to $\mathrm{Tr_B}\rho$, where ρ is the projector on $\boldsymbol{\Psi}_{\mathrm{AB}}$. Obviously, when the state of the system is a product state ($\boldsymbol{\Psi}_{\mathrm{AB}} = \psi_{\mathrm{A}} \otimes \psi_{\mathrm{B}}$), ρ_{A} is the projector on ψ_{A}, and we have that $\rho_{\mathrm{A}} = \rho_{\mathrm{A}}^2$, and $\mathrm{Tr}\rho_{\mathrm{A}} = \mathrm{Tr}\rho_{\mathrm{A}}^2 = 1$.

As we mentioned before, $\mathrm{Tr}\rho_{\mathrm{A}} - \mathrm{Tr}\rho_{\mathrm{A}}^2$ provides a good measure of the degree of the entanglement of the full system, in the vicinity of product states.

If the Schmidt bi-orthogonal decomposition of the state $\boldsymbol{\Psi}_{\mathrm{AB}}$ is equal to $\Sigma_{i:1\ldots \min(\mathrm{d_A},\mathrm{d_B})}\alpha_i'|\psi_{\mathrm{A}}^{i'}\rangle \otimes |\phi_{\mathrm{B}}^{i'}\rangle$, then it is easy to check that $\rho_{\mathrm{A}} = \Sigma_{i:1\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i'|^2|\psi_{\mathrm{A}}^{i'}\rangle\langle\psi_{\mathrm{A}}^{i'}|$, $\mathrm{Tr}\rho_{\mathrm{A}} = \Sigma_{i:1\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i'|^2 = 1$ by normalisation and $\mathrm{Tr}\rho_{\mathrm{A}}^2 = \Sigma_{i:1\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i'|^4 \leq (\mathrm{Tr}\rho_{\mathrm{A}})^2 = 1^2 = 1$. The last inequality is saturated for product states only. Note that $\mathrm{Tr}\rho_{\mathrm{A}}^2 = \mathrm{Tr}\rho_{\mathrm{B}}^2$, which shows that this parameter expresses properties of the system considered as a whole, as it must be when we are dealing with entanglement. Obviously $\mathrm{Tr}\rho_{\mathrm{A}}^2(t+\delta t) = |\alpha_1|^4 + \Sigma_{i:2\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i|^4$ and $\Sigma_{i:2\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i|^4 \leq (\Sigma_{i:2\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i|^2)^2$.

But $|\alpha_1|^4 = (1 - \Sigma_{i:2\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i|^2)^2 = (1 - \frac{\delta t^2}{\hbar^2}\Sigma_{i:2\ldots \mathrm{d_A};j:2\ldots \mathrm{d_B}}|H_{i1j1}|^2 + \tau(\delta t^3))^2$ and $(\Sigma_{i:2\ldots \min(\mathrm{d_A},\mathrm{d_B})}|\alpha_i|^2)^2 = (\frac{\delta t^2}{\hbar^2}\Sigma_{i:2\ldots \mathrm{d_A};j:2\ldots \mathrm{d_B}}|H_{i1j1}|^2 + \tau(\delta t^3))^2 = \tau(\delta t^4)$ in virtue of (23), so that $\mathrm{Tr}\rho_{\mathrm{A}}^2(t+\delta t) = 1 - 2\cdot\frac{\delta t^2}{\hbar^2}\Sigma_{i:2\ldots \mathrm{d_A};j:2\ldots \mathrm{d_B}}|H_{i1j1}|^2 + \tau(\delta t^3)$ for δt small enough, which proves that, short enough for time increments, the (quadratic in time) squared norm of the biorthogonal component that is generated during the evolution is equal to one halve of the increase of the linear entropy production. The first temporal derivative of this increase is always equal to zero for product states. This reflects an interesting geometrical property: in comparison to other states, product states are global minima of the linear entropy (this is also true for the Shannon-von Neumann entropy) [3].

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[3] It can be shown by direct computation that, when the state of the system is a product state ($\boldsymbol{\Psi}_{\mathrm{AB}}(t) = \psi_{\mathrm{A}}(t) \otimes \psi_{\mathrm{B}}(t)$), then the identity $\frac{\mathrm{d}\mathrm{Tr}\rho_{\mathrm{A}}^2}{\mathrm{d}t}(t) = 0$ is necessarily satisfied, independently of the form of the Hamiltonian H_{AB}. This explains why no term of the first order in time appears in the previous development. This is a good point in the favour of the PS criterion, where it is assumed that such states exhibit a certain structural stability.